VARIATIONAL INEQUALITY PROBLEM INVOLVING MULTIVALUED NONEXPANSIVE MAPPING IN CAT(0) SPACES

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Abstract. In this paper, we introduce a new type of variational inequality problem (VIP) involving nonself multivalued mappings in CAT(0) spaces. We show that this VIP problem admit a solution under suitable conditions. We also perform the convergence analysis of introduced VIP via proximal multivalued Picard-S iteration. We finish our paper with some convergence theorems for new iteration scheme and system of variational inequalities associated with finite family of nonself multivalued nonexpansive mappings.

1. Introduction and Preliminaries

Many real life problem are moulded in the form of variational inequality problems (VIPs). VIPs has many applications in different fields such as optimization theory, economic equilibrium, game theory, mechanics and some others. Due to its wide applications, many authors studied VIPs associated with various mappings. After Hartman-Stampacchia theorem (see [1], [2]), this topic has become an independent research topic and still continues to attract attention of researchers (see [3]–[8]). However, most of the studies done in linear spaces like Hilbert spaces even though most of the real life problems arise from nonlinear structures. Therefore, to study VIPs in nonlinear structures like geodesic spaces is really important and it is gaining attention of the researchers. In 2015, Khatibzadeh and Ranjbar introduced VIP problem associated with single valued nonexpansive mappings in CAT(0) space. They proved some existence and convergence results for the solution of the problem associated with non-self operator in CAT(0) spaces.

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Motivated by this study, we introduce a new variational inequality problem (VIP) associated with non-self multivalued nonexpansive mappings in CAT(0) spaces. We show that the VIP problem admit a solution under suitable condition. Further, we show that proximal multivalued Picard-S iteration converges to the solution of the problem as well as fixed point of appropriate mapping. To follow this study, we introduce system of variational inequalities associated with finite family of non-self multivalued nonexpansive mappings in CAT(0) spaces and prove that modified proximal multivalued Picard-S iteration is Δ-convergent and strong convergent to a common solution of the system.

To make our paper self contained, lets collect some relevant and needed background material. Let (X, d) be a metric space then the family of nonempty, closed and convex subsets of X, the family of nonempty compact and convex subsets of X, the family of nonempty closed and bounded convex subsets of X will be denoted by $C(X)$, $KC(X)$, $CB(X)$, respectively. A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = \operatorname{dist}(x,K) = \inf\{d(x,y) : y \in K\}.$$

We shall denote by $PB(K)$, the family of nonempty bounded proximinal subsets of K.

The set of all proximinal point from x to K is denoted by

$$P_K(x) = \big\{y \in K : d(x,y) = d(x,K)\big\}.$$

This defines a mapping P_K from X into 2^K and is called the metric projection to K. The metric projection mapping is also known as the nearest point projection mapping, proximity mapping and best approximation operator. Let H be a Hausdorff metric on $CB(X)$, defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\right\},$$

where $d(x,B) = \inf\{d(x,y) : y \in B\}$. A multivalued mappings $T : X \to CB(X)$ is called nonexpansive if for all $x, y \in X$

$$H(Tx,Ty) \leq d(x,y)$$

is satisfied. A point is called fixed point of T if $x \in Tx$ and the set of all fixed points of T is denoted by $F(T)$.

In 2005, Sastry and Babu [8] defined Ishikawa iteration scheme for multivalued mappings. Let $T : K \to PB(K)$ a multivalued mapping and fix $p \in F(T)$. Then the sequence of Ishikawa iteration is defined as follows:

Choose $x_1 \in K$,

$$y_n = \beta_n z_n + (1 - \beta_n) x_n, \quad \{\beta_n\} \subset [0, 1],$$

where $z_n \in Tx_n$ such that $\|z_n - p\| = d(p, Tx_n)$ and

$$x_{n+1} = \alpha_n z_n' + (1 - \alpha_n) x_n, \quad \{\alpha_n\} \subset [0, 1],$$

where $z_n' \in Ty_n$ such that $\|z_n' - p\| = d(p, Ty_n)$.

Sastry and Babu [8] proved that Ishikawa iteration scheme for a multivalued nonexpansive mapping T (with a fixed point p) converges to a fixed point p of T under suitable conditions. In 2007, Panyanak [9] extended the results of Sastry and Babu to uniformly convex Banach space for multivalued nonexpansive mappings. Panyanak also modified the iteration scheme of Sastry and Babu and posed the question of convergence of this scheme.

In 2009, Song and Wang [10] pointed out the gap in the result of Panyanak [9]. They removed the gap and offered a partial answer to the question raised by Panyanak.

Simultaneously, Shahzad and Zegeye [11] extended the results of Sastry and Babu [8], Song and Wang [10] and Panyanak [9] to quasi nonexpansive multivalued mappings beside relaxing the end point condition and compactness of the domain by using the following modified iteration scheme which in fact offers an affirmative answer to a question raised by Panyanak [9] even in a more general setting.

Let (X, d) be a metric space, $x, y \in X$ and $C \subseteq X$ nonempty subset. A geodesic path joining x and y is a map $c : [0, t] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(t) = y$ and $d(c(r), c(s)) = |r - s|$ for all $r, s \in [0, t]$. The image of c, $c([0, t])$ is called geodesic segment from x to y and if it is unique, then it is denoted by $[x, y]$. $z \in [x, y]$ if and only if for an $\lambda \in [0, 1]$ such that $d(z, x) = (1 - \lambda) d(x, y)$ and $d(z, y) = \lambda d(x, y)$. The point z is denoted by $z = (1 - \lambda) x \oplus \lambda y$. If for every $x, y \in X$ there is a geodesic path then (X, d) called geodesic space and uniquely geodesic space if that geodesic path is unique for any pair x, y. A subset $C \subseteq X$ is called convex if it contains all geodesic segment joining any pair of points in it.

A geodesic space is called CAT(0) space if and only if the inequality (**CN***)

$$d^2\left(x,(1-\lambda)\,y\oplus\lambda z\right)\leq(1-\lambda)\,d^2\left(x,y\right)+\lambda d^2\left(x,z\right)-\lambda\left(1-\lambda\right)d^2\left(y,z\right)$$

satisfied for every $x,y,z\in X$, $\lambda\in[0,1]$.

Proposition 1.1 ([13]). *Let (X,d) be a* CAT(0) *space. Then, for any $x,y,z\in X$ and $\lambda\in[0,1]$, we have*

$$d\left((1-\lambda)\,x\oplus\lambda y,z\right)\leq(1-\lambda)\,d\left(x,z\right)+\lambda d\left(y,z\right).$$

Let $\{x_n\}$ be a bounded sequence on X and $x\in X$. Then, with setting

$$r\left(x,\{x_n\}\right)=\limsup_{n\to\infty}d\left(x,x_n\right)$$

the asymptotic radius of $\{x_n\}$ is defined by

$$r\left(\{x_n\}\right)=\inf\big\{r\left(x,\{x_n\}\right):x\in X\big\},$$

the asymptotic radius of $\{x_n\}$ with respect to $K\subseteq X$ is defined by

$$r_K\left(\{x_n\}\right)=\inf\big\{r\left(x,\{x_n\}\right):x\in K\big\},$$

and the asymptotic center of $\{x_n\}$ is defined by

$$A\left(\{x_n\}\right)=\big\{x\in X:r\left(x,\{x_n\}\right)=r\left(\{x_n\}\right)\big\},$$

and let $\omega_w\left(x_n\right):=\cup A\left(\{x_n\}\right)$, where union is taken over all subsequences of $\{x_n\}$.

Definition 1.2 ([15]). A sequence $\{x_n\}\subset X$ is said to be Δ-convergent to $x\in X$ if x is the unique asymptotic center of all subsequence $\{u_n\}$ of $\{x_n\}$, i.e. $\omega_w\left(x_n\right):=\cup A\left(\{x_n\}\right)=\{x\}$. In this case we write $\Delta\text{-}\lim_n x_n=x$.

Lemma 1.3 ([13]). (i) *Every bounded sequence in a complete* CAT(0) *space has a Δ-convergent subsequence.*

(ii) *If K is a closed convex subset of a complete* CAT(0) *and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 1.4 ([13]). *If $\{x_n\}$ is a bounded sequence in X with $A\left(\{x_n\}\right)=\{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A\left(\{u_n\}\right)=u$ and the sequence $\{d\left(x_n,u\right)\}$ converges, then $x=u$.*

Theorem 1.5 ([14]). *Let X be a bounded, complete and uniformly convex metric space. If T is a multivalued nonexpansive mapping which assigns to each point of X a nonempty compact subset of X, then T has a fixed point in X.*

Gursoy and Karakaya [20] (see also [21]) introduced Picard-S iteration as follows:

$$\begin{aligned} x_{n+1} &= Ty_n, \\ y_n &= (1-\alpha_n)\, Tx_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)\, x_n + \beta_n T x_n, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. They showed that it converges to fixed point of contraction mappings faster than Ishikawa, Noor, SP, CR, S and some other iterations. Also, they used it to solve certain delay differential equations.

Now, we define multivalued version of Picad-S iteration in CAT(0) spaces. Let K nonempty, closed and convex subset of a CAT(0) space X and $T : K \to C(K)$ a mapping. Then, for any $x_0 \in K$ the proximal multivalued Picard-S iteration is defined as follows:

$$\begin{aligned} x_{n+1} &= P_K(u_n), \\ y_n &= P_K((1-\alpha_n)\, w_n \oplus \alpha_n v_n), \\ z_n &= P_K((1-\beta_n)\, x_n \oplus \beta_n w_n), \end{aligned} \tag{1.1}$$

where $u_n \in Ty_n$, $v_n \in Tz_n$ and $w_n \in Tx_n$ P_K is a metric projection, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

It is well known fact that in a complete CAT(0) space, the metric projection $P_K(x)$ of x onto a nonempty, closed and convex subset K is singleton and nonexpansive. Berg and Nikolaev [19] introduced the concept of quasi-linearization which in an analogue of inner-product in CAT(0) space as follows: For any $a, b \in X$, $\overrightarrow{ab}$ as a vector in X, quasi-linear mapping is defined as

$$\langle\, , \rangle : (X \times X) \times (X \times X) \to \mathbb{R},$$

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\Big[d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\Big]$$

for all $a, b, c, d \in X$ which satisfies following properties

$$\begin{aligned} \langle \overrightarrow{ab}, \overrightarrow{ab} \rangle &= d^2(a,b), \\ \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle &= -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle, \\ \langle \overrightarrow{ab}, \overrightarrow{ab} \rangle &= \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle, \\ \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle &= d(a,b)d(c,d) \end{aligned}$$

for all $a, b, c, d, e \in X$. The last properties is known as a Cauchy-Schwarz inequality and it is a characterization of CAT(0) space: A geodesic metric space is a CAT(0) *if and only if* it satisfies Cauchy-Schwarz inequality.

Lemma 1.6 ([19]). *Let X be a* CAT(0) *and K be a nonempty and convex subset of X, $x \in X$ and $u \in K$. Then $u = P_K(x)$ if and only if*

$$\langle \overrightarrow{xu}, \overrightarrow{yu} \rangle \leq 0 \text{ for all } y \in K.$$

For a nonempty closed and convex subset K of a real Hilbert space X, an operator $A : K \to 2^X$ is called monotone if and only if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for all $x, y \in X$, $x^* \in Ax$ and $y^* \in Ay$. The variational inequality associated with monotone operator A is finding $(u, x)_{u \in Ax}$ such that

$$\langle u, y - x \rangle \geq 0 \quad \text{for all} \quad y \in K.$$

The VIPs associated with monotone operators have several applications in applied mathematics. For further details and applications of VIPs, one can consult Kinderlehrer and Stampacchia (see [1, 2]).

In 2015, Khatibzadeh and Ranjbar [18] introduced the concept of the variational inequality associated with the nonexpansive mapping T in CAT(0) space as follows:

$$\textit{Find } x \in K \textit{ such that } \left\langle \overrightarrow{Txx}, \overrightarrow{xy} \right\rangle \geq 0 \quad \textit{for all} \quad y \in K.$$

They proved some existence and convergence results for this problem.

The aim of this is to introduce the concept of variational inequality associated with a non-self multivalued nonexpansive mapping T which run as follows:

$$\textit{Find } (u, x)_{u \in Tx} \textit{ such that } \left\langle \overrightarrow{ux}, \overrightarrow{xy} \right\rangle \geq 0 \quad \textit{for all} \quad y \in K. \tag{1.2}$$

We prove some existence and convergence results for this problem.

2. Results Concerning Existence of A Solution

Throughout this section, let us assume that K be a nonempty, closed and convex subset of a complete $CAT(0)$ space X.

Definition 2.1. If K is also bounded subset of X and $T : K \to C(X)$, then the projection $P_K T$ of multivalued mapping T onto K is defined by

$$\begin{aligned} P_K^* T(x) &= \bigcup_{x' \in Tx} \{P_K(x')\} = \{P_K(x') : x' \in Tx\} \\ &= \big\{v \in K : d(x', v) = D(x', K),\ x' \in Tx\big\}, \end{aligned}$$

where P_K is metric projection and $D(x', K) = \inf_{v' \in K} d(x', v')$.

Lemma 2.2. *$P_K^* T(x)$ is a multivalued nonexpansive mapping from K to 2^K.*

Proof. Since K is closed, convex and bounded, $P_K^*(Tx) \subset K$. By the nonexpansiveness of P_K, we have

$$\begin{aligned} H(P_K^*(Tx), P_K^*(Ty)) = \max \Bigg\{ & \sup_{P_K(x') \in P_K^* Tx} \inf_{P_K(y') \in P_K^* Ty} d(P_K(x'), P_K(y'), \\ & \sup_{P_K(y') \in P_K^* Ty} \inf_{P_K(x') \in P_K^* Tx} d(P_K(y'), P_K(x') \Bigg\} \\ \le \max & \left\{ \sup_{x' \in Tx} \inf_{y' \in Ty} d(x', y'), \sup_{y' \in Ty} \inf_{x' \in Tx} d(y', x' \right\} \\ = H & (Tx, Ty) \\ \le d & (x, y). \end{aligned}$$

$\square$

Lemma 2.3. *If T is compact valued then P_K^* is compact valued.*

Proof. Let $(v_n) \subset P_K^* T(x)$ be a sequence. Then there is a sequence $(x'_n) \subset Tx$ such that $v_n = P_K(x'_n)$ for all $n \in \mathbb{N}$. Since T is compact valued then (x'_n) has a convergent subsequence (x'_{n_k}) with $\lim_{k\to\infty} x'_{n_k} = z \in Tx$ and since

$$d(P_K(x'_{n_k}), P_K(z)) \le d(x'_{n_k}, z)$$

for all $k \in \mathbb{N}$, we get that the sequence $(v_n) = (P_K(x'_n))$ has a convergent subsequence $(v_{n_k}) = (P_K(x'_{n_k}))$ with

$$\lim_{k\to\infty} (P_K(x'_{n_k})) = P_K(z) \in P_K^* T(x).$$

Hence, $P_K^* Tx$ is compact. □

Theorem 2.4. *If $T : K \to KC(X)$, then there exists a solution $(u, x)_{u\in Tx}$ of the variational inequality* (1.2).

Proof. Since X is uniformly convex and T is compact valued, then, by Theorem 1.5, $P_K T$ has a fixed point $p \in P_K T(p) \subset K$. By the definition of $P_K T$, there exists a $p' \in Tp$ such that $p = P_K(p')$. By Lemma 1.6, we have $\langle p'p, yp\rangle \leq 0$ for all $y \in K$. Hence, we have

$$\langle p'p, yp\rangle \geq 0 \quad \text{for all } y \in K,$$

in which $p' \in Tp$. Therefore, $(p', p)_{p'\in Tp}$ is a solution of the problem (1.2). □

Example 2.5. Let $X = \mathbb{R}^2$ and $K = \big\{(x, y) : 0 \leq x, 0 \leq y, x^2 + y^2 \leq 1\big\}$. Define an operator $T : K \to KC(X)$ by

$$T(x, y) = B_1[(\sin x - 1, \sin y - 1)].$$

Then T is a nonexpansive mapping without a fixed point. Also, $(-1, -1)$ is in $T(0, 0)$ so that $((-1, -1), (0, 0))$ is a solution of problem (1.2).

Theorem 2.6. *If $x \in \operatorname{int}(K)$ and $(u, x)_{u\in Tx}$ is a solution of* (1.2)*, then $x \in F(T)$, i.e., $u = x$.*

Proof. There exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset K$. Let us take $t \in (0, 1)$ such that $tx \oplus (1 - t)u \in B(x, \varepsilon)$, that is, $d(x, tx \oplus (1 - t)u) = (1 - t)d(x, u) < \varepsilon$. Since $B(x, \varepsilon) \subset K$, then $tx \oplus (1 - t)u \in K$ and $d(u, tx \oplus (1 - t)u) = td(u, x)$. Hence, we have

$$\begin{aligned}
0 &\leq 2\Big\langle \overrightarrow{ux}, \overrightarrow{x(tx \oplus (1-t)u)}\Big\rangle \\
&= d^2(u, tx \oplus (1 - t)u) - d^2(x, u) - d^2(x, tx \oplus (1 - t)u) \\
&= t^2 d^2(x, u) - d^2(x, u) - (1 - t)^2 d^2(x, u) \\
&= 2(t^2 - 1)d(x, u) \leq 0,
\end{aligned}$$

which implies $2(t-1)d(x,u)=0$. Since $t\in(0,1)$, $d(x,u)=0$. Hence, we have $u=x\in Tx$. $\square$

Example 2.7. Let $X=\mathbb{R}^2$, $K=[0,\pi]^2$ and $T:K\to KC(X)$ be defined by

$$T(x,y)=\Big\{(\sin x+2\pi,\sin y+2\pi),\Big(\cos x+\frac{\pi}{2},\cos y+\frac{\pi}{2}\Big),$$
$$(\arctan x-2\pi,\arctan y-2\pi)\Big\}$$

Then T is a nonexpansive mapping with the fixed point $(\frac{\pi}{2},\frac{\pi}{2})$. Also $(\frac{\pi}{2},\frac{\pi}{2})$ is in $T(\frac{\pi}{2},\frac{\pi}{2})$ so that $((\frac{\pi}{2},\frac{\pi}{2}),(\frac{\pi}{2},\frac{\pi}{2}))$ is a solution of the problem (1.2).

If K is not bounded, then (1.2) does not always has a solution. However, let $o\in X$ be arbitrary and set $K_r=K\cap B(o,r)$. If $K_r\neq\emptyset$, then, by Theorem 2.4, there exists an $x_r\in K_r$ such that $(u_r,x_r)_{u_r\in Tx}$ is a solution of the problem

$$\langle\overrightarrow{u_rx_r},\overrightarrow{x_ry}\rangle\geq 0\ \text{ for all }\ y\in K_r. \tag{2.1}$$

Theorem 2.8. *The problem* (1.2) *has a solution if and only if there exists an* $r>0$ *such that the solution of* (2.1)*, that is,* $(u_r,x_r)_{u_r\in Tx_r}$ *with* $x_r\in K_r$ *satisfies* $d(o,x_r)<r$.

Proof. If the problem 1.2 has a solution $(u,x)_{x\in Tx}$, then $(u,x)_{x\in Tx}$ is a solution of the problem (2.1) and thus $d(o,x)<r$ is satisfied. Now, assume that there exists an $r>0$ such that $(u_r,x_r)_{u_r\in Tx_r}$ with $x_r\in K_r$ satisfies $d(o,x_r)<r$. Let $y\in K$ be arbitrary. We can choose $t\in(0,1)$ such that $(1-t)x_r\oplus ty\in B(o,r)$, that is, $(1-t)x_r\oplus ty\subset K_r$ and $d(x_r,(1-t)x_r\oplus ty)=td(x_r,y)$. Then we have

$$\begin{aligned}
0&\leq 2\langle\overrightarrow{u_rx_r},\overrightarrow{x_r((1-t)x_r\oplus ty)}\rangle\\
&=d^2(u_r,(1-t)x_r\oplus ty)-d^2(x_r,u_r)-d^2(x_r,(1-t)x_r\oplus ty)\\
&\leq(1-t)d^2(u_r,x_r)+td^2(u_r,y)-t(1-t)d^2(x_r,y)-d^2(x_r,u_r)-t^2d^2(x_r,y)\\
&=2t(d^2(u_r,y)+d^2(x_r,x_r)-d^2(u_r,x_r)-d^2(x_r,y))\\
&=2t\langle\overrightarrow{u_rx_r},\overrightarrow{x_ry}\rangle.
\end{aligned}$$

Hence,

$$\langle\overrightarrow{u_rx_r},\overrightarrow{x_ry}\rangle\geq 0\ \text{ for all }\ y\in K,$$

that is, $(u_r, x_r)_{u_r \in Tx_r}$ is a solution of the problem (1.2). $\square$

Theorem 2.9. *Let $T : K \to KC(X)$ and $o \in X$ be fixed. If there exist $x_0 \in K$ and $u_0 \in Tx_0$ such that*

$$\frac{\langle \overrightarrow{ux}, \overrightarrow{x_0x} \rangle - \langle \overrightarrow{u_0x_0}, \overrightarrow{x_0x} \rangle}{d(x, x_0)} \to \infty \quad as \quad d(x, o) \to \infty$$

in which $u \in Tx$ such that $d(x, u) = d(x, Tx)$, then the problem (1.2) has a solution.

Proof. Let $r, M \in \mathbb{R}$ such that $d(u_0, x_0) < M$, $d(x_0, o) < r$ and

$$\langle \overrightarrow{ux}, \overrightarrow{x_0x} \rangle - \langle \overrightarrow{u_0x_0}, \overrightarrow{x_0x} \rangle \geq Md(u_0, x_0)$$

for all $x \in K$ with $d(x, o) \geq r$. Then, for all $x \in K$ with $r = d(x, o)$, we have

$$\begin{aligned} \langle \overrightarrow{ux}, \overrightarrow{x_0x} \rangle &\geq \langle \overrightarrow{u_0x_0}, \overrightarrow{x_0x} \rangle + Md(u_0, x_0) \\ &\geq -d(u_0, x_0)d(x_0, x) + Md(u_0, x_0) \\ &\geq (M - d(x_0, x))d(u_0, x_0) \\ &\geq (M - d(x_0, x))(d(x, o) - d(x_0, o)). \end{aligned}$$

If $(u_r, x_r)_{u_r \in Tx_r}$ is a solution of the problem (2.1), then we have

$$\langle \overrightarrow{u_rx_r}, \overrightarrow{x_0x_r} \rangle = -\langle \overrightarrow{u_rx_r}, \overrightarrow{x_rx_0} \rangle \leq 0,$$

which implies $d(x_r, o) < r$. Hence, by Theorem 2.8, the problem (1.2) has a solution. $\square$

3. Results Concerning Convergence Analysis

In this section, it is assumed that X is a complete CAT(0) and K is a nonempty, closed and convex subset of X. Since a CAT(0) space is also a hyperbolic space then the following is still true:

Lemma 3.1 ([22]). *Let (X, d, W) be a uniformly convex hyperbolic space with modulus of uniform convexity δ. If $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq r$ for any $r > 0$, $\varepsilon \in (0, 2)$, $\lambda \in [0, 1]$, and $a, x, y \in X$, then we have*

$$d((1 - \lambda)x \oplus \lambda y, z) \leq (1 - 2\lambda(1 - \lambda)\delta(r, \varepsilon))r.$$

Proposition 3.2 ([23]). *Assume that X is a CAT(0) space. Then, X is uniformly convex and $\delta(r, \varepsilon) = \varepsilon^2/8$ is a modulus of uniform convexity.*

Lemma 3.3 ([23]). *Let X be a complete* CAT(0) *space with modulus of convexity $\delta(r,\varepsilon)$ and let $x \in E$. Suppose that $\delta(r,\varepsilon)$ increases with r (for a fixed ε), $\{t_n\}$ is a sequence in $[b,c]$ for some $b,c \in (0,1)$, $\{x_n\}$ and $\{y_n\}$ are sequences in X such that* $\limsup_{n\to\infty} d(x_n,x) \leq r$, $\limsup_{n\to\infty} d(y_n,x) \leq r$ *and* $\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$ *for some $r \geq 0$. Then we have* $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Theorem 3.4. *If $T : K \to KC(X)$ is a nonexpansive mapping, $\{x_n\}$ is a bounded sequence in K with* $\Delta\text{-}\lim_{n\to\infty} x_n = z$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, *then $z \in K$ and $z \in T(z)$.*

Proof. By Lemma 1.3, we have $z \in K$. We can find a sequence $\{y_n\}$ such that $y_n \in Tx_n$, $d(x_n,y_n) = d(x_n,Tx_n)$ which implies $\lim_{n\to\infty} d(x_n,y_n) = 0$. We can also find a sequence $\{z_n\}$ in Tz such that $d(y_n,z_n) = d(y_n,Tz)$. Since Tz is compact, there exists a convergent subsequence $\{z_{n_i}\}$ of $\{z_n\}$, say $\lim_{i\to\infty} z_{n_i} = u \in Tz$. Now, we have

$$\begin{aligned} d(x_{n_i},u) &\leq d(x_{n_i},y_{n_i}) + d(y_{n_i},z_{n_i}) + d(z_{n_i},u) \\ &\leq d(x_{n_i},y_{n_i}) + d(y_{n_i},Tz) + d(z_{n_i},u) \\ &\leq d(x_{n_i},y_{n_i}) + H(Tx_{n_i},Tz) + d(z_{n_i},u), \end{aligned}$$

which implies

$$\limsup_{i\to\infty} d(x_{n_i},u) \leq \limsup_{i\to\infty} H(Tx_{n_i},Tz)$$

and $\Delta\text{-}\lim_{i\to\infty} x_{n_i} = z$. By the nonexpansiveness of T, we have $H(Tx_{n_i},Tz) \leq d(x_{n_i},z)$, which gives

$$\limsup_{i\to\infty} d^2(x_{n_i},u) \leq \limsup_{i\to\infty} H^2(Tx_{n_i},Tz) \leq \limsup_{i\to\infty} d^2(x_{n_i},z)$$

which further yields $z = u \in Tz$. □

Lemma 3.5. *If $T : K \to KC(X)$ is a nonexpansive mapping and $\{x_n\}$ is a bounded sequence in K with* $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ *and $\{d(x_n,p)\}$ converges for all $p \in F(T)$, then $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ includes exactly one point.*

Proof. Assume that $u \in \omega_w(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. Also, by Lemma 1.3, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ with $\Delta\text{-}\lim_{n\to\infty} v_n = v \in K$. Now, by Theorem 3.4, we have $v \in F(T)$. Therefore, by Lemma 1.4, we conclude that $u = v$ which yields $\omega_w(x_n) \subseteq F(T)$.

Now, we consider a subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq F(T)$, $\{d(x_n, u)\}$ is convergent. Hence, by the conclusion of Lemma 1.4, we have $x = u$. This means that $\omega_w(x_n)$ includes exactly one point. $\square$

Theorem 3.6. *If $T : K \to C(X)$ is a nonexpansive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$ and $\{x_n\}$ is a sequence in K defined by* (1.1) *with $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\{x_n\}$ is bounded, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, and $\{d(x_n, p)\}$ converges for all $p \in F(T)$.*

Proof. Let $p \in F(T)$, then $T(p) = \{p\}$ which implies that

$$d(u, p) = d(u, Tp) \leq H(Tx, Tp) \leq d(x, p).$$

for all $x \in K$ and $u \in Tx$. Hence, $d(u, p) \leq d(x, p)$. Now, we have

$$\begin{aligned} d(z_n, p) &= d(P_K((1-\beta_n)x_n \oplus \beta_n w_n), p) \\ &\leq d((1-\beta_n)x_n \oplus \beta_n w_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(w_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p), \end{aligned}$$

$$\begin{aligned} d(y_n, p) &= d(P_K((1-\alpha_n)w_n \oplus \alpha_n v_n, p) \\ &\leq d((1-\alpha_n)w_n \oplus \alpha_n v_n, p) \\ &\leq (1-\alpha_n)d(w_n, p) + \alpha_n d(v_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(z_n, p) \\ &\leq d(x_n, p), \end{aligned}$$

and $d(x_{n+1}, p) = d(P_K(u_n), P_K(p)) \leq d(u_n, p) \leq d(y_n, p)$.

Therefore, $d(x_{n+1}, p) \leq d(y_n, p) \leq d(x_n, p)$ which implies that

$$\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(y_n, p) = k, \quad k \in \mathbb{R}.$$

Since $d(w_n, p) \leq d(x_n, p)$ and $d(v_n, p) \leq d(z_n, p) \leq d(x_n, p)$, we have that

$$\limsup_{n\to\infty} d(w_n, p) \leq k, \limsup_{n\to\infty} d(v_n, p) \leq k,$$

and

$$\begin{aligned}
d(y_n, p) &= d(P_K((1-\alpha_n)w_n \oplus \alpha_n v_n, p) \\
&\le d((1-\alpha_n)w_n \oplus \alpha_n v_n, p) \\
&\le (1-\alpha_n)d(w_n, p) + \alpha_n d(v_n, p) \\
&\le (1-\alpha_n)d(x_n, p) + \alpha_n d(z_n, p), \\
&\le d(x_n, p),
\end{aligned}$$

which implies that $\lim_{n\to\infty} d((1-\alpha_n)w_n \oplus \alpha_n v_n, p) = k$. So, by the conclusion of Lemma 3.3, we get $\lim_{n\to\infty} d(w_n, v_n) = 0$. Now, we have the following estimates:

$$\begin{aligned}
d(y_n, p) &= d(P_K((1-\alpha_n)w_n \oplus \alpha_n v_n, p) \\
&\le d((1-\alpha_n)w_n \oplus \alpha_n v_n, p) \\
&\le (1-\alpha_n)d(w_n, p) + \alpha_n d(v_n, p) \\
&\le (1-\alpha_n)(d(w_n, v_n) + d(v_n, p)) + \alpha_n d(v_n, p) \\
&\le (1-\alpha_n)d(w_n, v_n) + d(v_n, p),
\end{aligned}$$

which gives $k \le \liminf_{n\to\infty} d(v_n, p)$.

Since $d(v_n, p) \le d(z_n, p) \le d(x_n, p)$,

$$\lim_{n\to\infty} d(z_n, p) = k.$$

By (**CN***) inequality, we have

$$\begin{aligned}
d^2(z_n, p) &= d^2(P_K((1-\beta_n)x_n \oplus \beta_n w_n), p) \\
&\le d^2((1-\beta_n)x_n \oplus \beta_n w_n, p) \\
&\le (1-\beta_n)d^2(x_n, p) + \beta_n d^2(w_n, p) - (1-\beta_n)\beta_n d^2(x_n, w_n) \\
&\le (1-\beta_n)d^2(x_n, p) + \beta_n d^2(x_n, p) - (1-\beta_n)\beta_n d^2(x_n, w_n) \\
&\le d^2(x_n, p) - (1-\beta_n)\beta_n d^2(x_n, w_n),
\end{aligned}$$

which implies that $(1-\beta_n)\beta_n d^2(x_n, w_n) \le d^2(x_n, p) - d^2(z_n, p)$.

Since $\lim_{n\to\infty}(d^2(x_n, p) - d^2(z_n, p)) = 0$ and $\liminf_n (1-\beta_n)\beta_n > 0$,

$$\lim_{n\to\infty} d(x_n, w_n) = 0.$$

Hence, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. $\square$

Theorem 3.7. *If $T : K \to KC(X)$ is a nonexpansive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$ and $\{x_n\}$ is a sequence in K defined by* (1.1) *with* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, *then $\{x_n\}$ is Δ-convergent to $p \in F(T)$ in which (p,p) is a solution of* (1.2)

Proof. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, $\{d(x_n,p)\}$ converges for all $p \in F(T)$. By Theorem 3.6, $\{x_n\}$ is a bounded sequence. It follows from Lemma 3.5 that $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ includes exactly one point $p \in F(T)$ in which (p,p) is a solution of (1.2). □

Theorem 3.8. *Let K be compact and $T : K \to C(X)$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. If $\{x_n\}$ is a sequence in K defined by* (1.1) *with* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, *then $\{x_n\}$ strongly converges to $q \in F(T)$ in which (q,q) is a solution of* (1.2).

Proof. By Theorem 3.6, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Since K is compact, there exists a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$, say $\lim_{i\to\infty} x_{n_i} = q$. Then, we have

$$\begin{aligned} d(q,Tq) &\leq d(q,x_{n_i}) + d(x_{n_i},Tx_{n_i}) + H(Tx_{n_i},Tq) \\ &\leq 2d(q,x_{n_i}) + d(x_{n_i},Tx_{n_i}). \end{aligned} \tag{3.1}$$

By passing to the limit on i in (3.1), we obtain $q \in Tq$. □

Example 3.9. Let $X = \mathbb{R}^2$ and $K = \{(x,y) : 0 \leq x, 0 \leq y, x^2+y^2 \leq 1\}$. Define an operator $T : K \to KC(X)$ by

$$T(x,y) = B_{\frac{|x-y|}{2\sqrt{2}}}\left[\left(\frac{x}{2},\frac{y}{2}\right)\right].$$

Then T is a nonexpansive mapping with $(0,0) \in F(T)$ and $T(0,0) = \{(0,0)\}$ which is also a solution of problem (1.2). The convergence behaviours of iteration scheme (1.1) for different choice of initial points are shown in Figure 1.

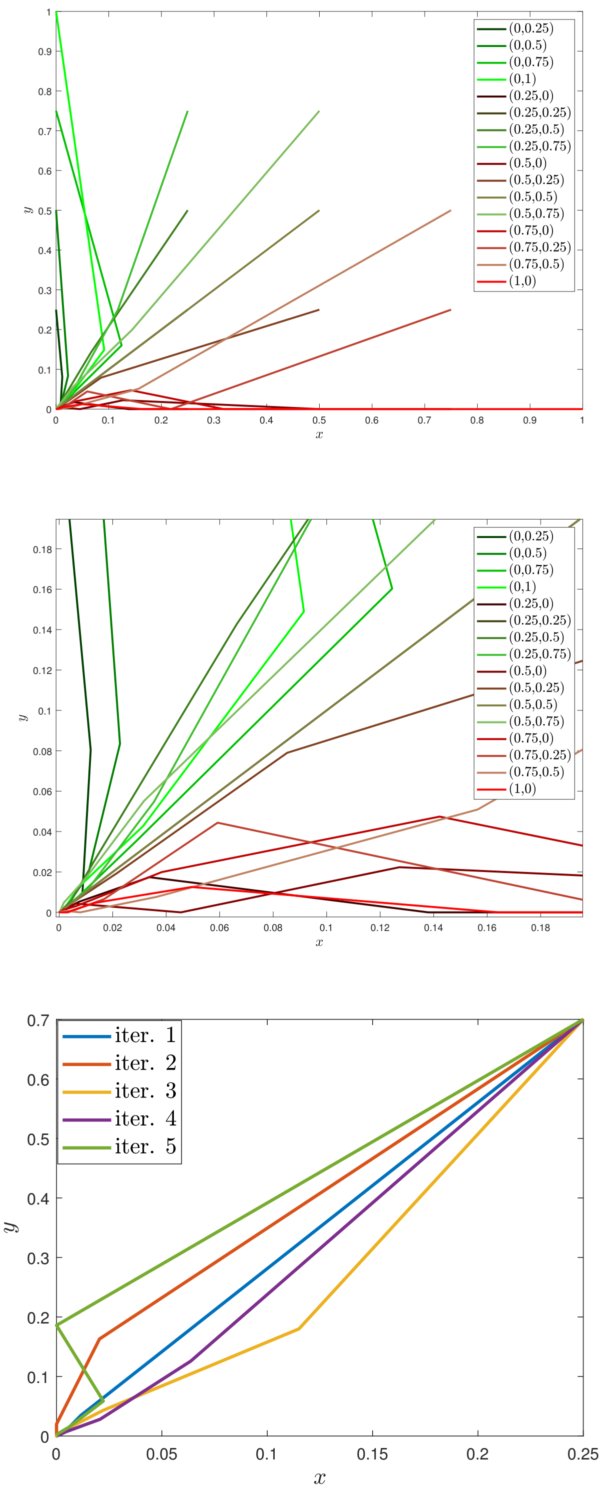


FIGURE 1. he convergence behaviours of iteration scheme (1.1) for different choice of initial points [graphs in $[0,1]^2$ (top) and enlarged part $[0,0.2]^2$ (middle)] and for the same initial point (down)

4. Common Solution of System of Variational Inequalities

Let $K_i \subset X$ be a finite family of nonempty, closed and convex subsets of CAT(0) space X with $\bigcap_{i=1}^{N} K_i \neq \emptyset$. If $T_i : K_i \to C(X)$ are multivalued mappings for $i = 1, \ldots, N$, then the system of variational inequalities problem is

$$\textit{Find } \ (u_i, x)_{u_i \in T_i x} \ \ \textit{such that } \ \left\langle \overrightarrow{u_i x}, \overrightarrow{xy} \right\rangle \geq 0 \ \ \textit{for all } \ y \in K_i, \ i = 1, \ldots, N. \tag{4.1}$$

It is obvious that problem (4.1) reduced to problem (1.2) for $N = 1$. The importance of studying problem (4.1) is underlying on fact that it is a unification of many problems. For example,

i) if we take $T_i = 0$ for all $i = 1, \ldots, N$, problem (4.1) reduced to the following convex feasibility problem

$$\textit{Find } \ x \in K = \bigcap_{i=1}^{N} K_i \,;$$

ii) if T_i, $i = 1, \ldots, N$ is a self operator and $K = \bigcap_{i=1}^{N} F(T_i)$, then problem (4.1) becomes a common fixed point problem.

Example 4.1. Let $X = \mathbb{R}^2$ and

$$K_n = \left\{ (x, y) : 0 \leq x, \ 0 \leq y, \ x^2 + y^2 \leq \sqrt[n]{\frac{1}{n}} \right\}$$

for all $n \in \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$. Define $T_n : K_n \to KC(X)$ by

$$T_n(x, y) = B_{\sqrt{n-1}}((\sin x - n, \sin y - n)) \ \text{ for all } \ n \in \{1, 2, \ldots, N\}.$$

Then T_n for all $n \in \{1, 2, \ldots, N\}$ is a nonexpansive mapping without a fixed point. Also, $(-n, -n) \in T_n(0, 0)$ so that $((-n, -n), (0, 0))$ is a solution of problem (4.1).

Let $K = \bigcap_{i=1}^{N} K_i \neq \emptyset$ and $x_1 \in K$. Then, the modified proximal multivalued Picard-S iteration is defined by

$$\begin{aligned} x_{n+1} &= P_K\left(\bigoplus_{i=1}^{N} \lambda_{n,i} u_{n,i}\right), \\ y_n &= P_K\left(\bigoplus_{i=1}^{N} \alpha_{n,i} w_{n,i} \oplus \bigoplus_{i=1}^{N} \beta_{n,i} v_{n,i}\right), \\ z_n &= P_K\left(\gamma_{n,0} x_n \oplus \bigoplus_{i=1}^{N} \gamma_{n,i} w_{n,i}\right), \quad n \geq 0, \end{aligned} \tag{4.2}$$

in which $u_{n,i} \in T_i y_n$, $w_{n,i} \in T_i x_n$, $v_{n,i} \in T_i z_n$, $\{\lambda_{n,i}\}$, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, $\{\gamma_{n,i}\}$ are real sequences in $[b,c] \subset (0,1)$ satisfying

$$\sum_{i=1}^{N} \lambda_{n,i} = 1, \quad \sum_{i=1}^{N} (\alpha_{n,i} + \beta_{n,i}) = 1, \quad \sum_{i=0}^{N} \gamma_{n,i} = 1.$$

We shall show that the iterative scheme defined by (4.2) is convergent to a common fixed point of family of non-self multivalued nonexpansive mappings $\{T_i\}_{i=1}^{N}$. This fixed point is also a solution of the system of variational inequalities problem (4.1).

Lemma 4.2 ([12])**.** *Let (X,d) be a complete* CAT(0) *space, $\{x_1, x_2, \ldots, x_n\} \subset X$, and $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset [0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$. Then, one has*

$$d\left(\bigoplus_{i=1}^{n} \lambda_i x_i, z\right) \leq \sum_{i=1}^{n} \lambda_i d(x_i, z)$$

for every $z \in X$.

We shall prove the following lemma before go further.

Lemma 4.3. *Let X be a complete $CAT(0)$ space with modulus of convexity $\delta(r,\varepsilon)$ and $x \in X$. Suppose that $\delta(r,\varepsilon)$ increases with r (for a fixed ε) and $\{t_{n,i}\}$ with $\sum_{i=1}^{N} t_{n,i} = 1$ is a sequence in $[b,c] \subset (0,1)$. Assume further that $\{x_{n,i}\}_{n=1}^{\infty}$, $i \in \{1,2,\ldots,N\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_{n,i}, x) \leq r \quad \text{and} \quad \lim_{n\to\infty} d\left(\bigoplus_{i=1}^{N} t_{n,i} x_{n,i}, x\right) = r$$

for some $r \geq 0$. Then, we have $\lim_{n\to\infty} d(x_{n,k}, x_{n,l}) = 0$ for $k,l \in \{1,2,\ldots,N\}$.

Proof. If $r = 0$, then the proof is clear. Now, let $r > 0$. Since $\limsup_{n\to\infty} d(x_{n,i}, x) \leq r$ for each $i = 1, 2, \ldots, N$, then, by Lemma 4.2, we have

$$\begin{aligned}
\lim_{n\to\infty} d\left(\bigoplus_{\substack{i=1,\\ i\neq m}}^{N} \frac{t_{n,i}}{1-t_{n,m}} x_{n,i}, x\right) &\leq \lim_{n\to\infty} \sum_{\substack{i=1,\\ i\neq m}}^{N} \frac{t_{n,i}}{1-t_{n,m}} d(x_{n,i}, x) \\
&\leq \lim_{n\to\infty} \sum_{\substack{i=1,\\ i\neq m}}^{N} \frac{t_{n,i}}{1-t_{n,m}} \left(\limsup_{n\to\infty} d(x_{n,i}, x)\right) \\
&\leq \lim_{n\to\infty} \sum_{\substack{i=1,\\ i\neq m}}^{N} \frac{t_{n,i}}{1-t_{n,m}} r = r,
\end{aligned}$$

for every $m = 1, 2, \ldots, N$.

Assume that $d(x_{n,k}, x_{n,l}) \nrightarrow 0$ for fixed $k, l \in \{1, 2, \ldots, N\}$ with $k \neq l$. Then, there exist subsequences $\{x_{n,k}\}$, $\{x_{n,l}\}$ of $\{x_n\}$ such that $inf_n d(x_{n,k}, x_{n,l}) > 0$. Since

$$\begin{aligned}
d\left(\bigoplus_{i=1}^{N} t_{n,i} x_{n,i}\, ,\, x_{n,m}\right) =& d\left((1-t_{n,m})\left[\bigoplus_{\substack{i=1\\ i\neq m}}^{N} \frac{t_{n,i}}{1-t_{n,m}} x_{n,i}\right] \oplus t_{n,m} x_{n,m},\, x_{n,m}\right) \\
\leq& (1-t_{n,m}) d\left(\bigoplus_{\substack{i=1\\ i\neq m}}^{N} \frac{t_{n,i}}{1-t_{n,m}} x_{n,i}, x_{n,m}\right) + t_{n,m} d(x_{n,m}, x_{n,m}) \\
=& (1-t_{n,m}) d\left(\bigoplus_{\substack{i=1\\ i\neq m}}^{N} \frac{t_{n,i}}{1-t_{n,m}} x_{n,i}, x_{n,m}\right),
\end{aligned}$$

$$\begin{aligned}
&0 < d(x_{n,k}, x_{n,l}) \\
&\leq d\left(\bigoplus_{i=1}^{N} t_{n,i} x_{n,i}, x_{n,k}\right) + d\left(\bigoplus_{i=1}^{N} t_{n,i} x_i, x_{n,l}\right) \\
&\leq (1-t_{n,k}) d\left(\bigoplus_{\substack{i=1\\ i\neq k}}^{N} \frac{t_{n,i}}{1-t_{n,k}} x_{n,i}, x_{n,k}\right) + (1-t_{n,l}) d\left(\bigoplus_{\substack{i=1\\ i\neq l}}^{N} \frac{t_{n,i}}{1-t_{n,l}} x_{n,i}, x_{n,l}\right).
\end{aligned}$$

As $t_{n,k}, t_{n,l} \in [b,c]$, by positivity of d, we have $d\left(\bigoplus_{i=1, i\neq k}^{N} \frac{t_{n,i}}{1-t_{n,k}} x_{n,i}, x_{n,k}\right) \nrightarrow 0$. Therefore, there exists a subsequence $\{x_{n,k}\}$ of $\{x_n\}$ for some $k = 1,2,\ldots,N$, such that $d\left(\bigoplus_{i=1}^{N} \frac{t_{n,i}}{1-t_{n,k}} x_{n,i}, x_{n,k}\right) > 0$ so that $d(x_{n,k}, x) \le r$, $d\left(\bigoplus_{i=1, i\neq k}^{N} \frac{t_{n,i}}{1-t_{n,k}} x_{n,k}, x\right) \le r$ and

$$\lim_{n\to\infty} d\left(\bigoplus_{i=1}^{N} t_{n,i}x_{n,i}, x\right) = \lim_{n\to\infty} d\left((1-t_{n,m})\left[\bigoplus_{\substack{i=1\\ i\neq k}}^{N} \frac{t_{n,i}}{1-t_{n,k}} x_{n,i}\right] \oplus t_{n,m}x_{n,m}, x\right) = r.$$

Now, we can apply Lemma 3.3 and hence, the proof is done. $\square$

From now on, it is assumed that X is a complete CAT(0) and $K = \bigcap_{i=1}^{N} K_i$ is a nonempty, closed and convex subset of X in which $K_i \subset X$ is a nonempty, closed and convex subset with $K = \bigcap_{i=1}^{N} K_i \neq \emptyset$ for all $i = 1,2,\ldots,N$.

Lemma 4.4. *Let $\{T_i\}_{i=1}^{N}$ be multivalued nonexpansive mappings from K to $CC(X)$ with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, $T_ip = \{p\}$ for all $p \in F$. Then, the sequence $\{x_n\}$ defined by* (4.2) *is bounded and the limit* $\lim_{n\to\infty} d(x_n, p)$ *exist for all $p \in F$.*

Proof. Let $p \in F$. Then, by the definition of $\{x_n\}$, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(P_K\left(\bigoplus_{i=1}^{N} \lambda_{n,i}u_{n,i}), p\right)\\
&\le \sum_{i=1}^{N} \lambda_{n,i} d(u_{n,i}, p)\\
&\le \sum_{i=1}^{N} \lambda_{n,i} d(u_{n,i}, T_ip)\\
&\le \sum_{i=1}^{N} \lambda_{n,i} H(T_iy_n, T_ip)\\
&\le \sum_{i=1}^{N} \lambda_{n,i} d(y_n, p)\\
&= d(y_n, p),
\end{aligned}$$

$$
\begin{aligned}
d(y_n, p) &= d\left(\bigoplus_{i=1}^{N} \alpha_{n,i} w_{n,i} \oplus \bigoplus_{i=1}^{N} \beta_{n,i} v_{n,i}, p\right) \\
&\leq \sum_{i=1}^{N} \alpha_{n,i} d(w_{n,i}, p) + \sum_{i=1}^{N} \beta_{n,i} d(v_{n,i}, p) \\
&\leq \sum_{i=1}^{N} \alpha_{n,i} d(w_{n,i}, T_i p) + \sum_{i=1}^{N} \beta_{n,i} d(v_{n,i}, T_i p) \\
&\leq \sum_{i=1}^{N} \alpha_{n,i} H(T_i x_n, T_i p) + \sum_{i=1}^{N} \beta_{n,i} H(T_i z_n, T_i p) \\
&\leq \sum_{i=1}^{N} \alpha_{n,i} d(x_n, p) + \sum_{i=1}^{N} \beta_{n,i} d(z_n, p),
\end{aligned}
$$

and

$$
\begin{aligned}
d(z_n, p) &= d\left(\gamma_{n,0} x_n \oplus \bigoplus_{i=1}^{N} \gamma_{n,i} w_{n,i}, p\right) \\
&\leq \gamma_{n,0} d(x_n, p) + \sum_{i=1}^{N} \gamma_{n,i} d(w_{n,i}, p) \\
&\leq \gamma_{n,0} d(x_n, p) + \sum_{i=1}^{N} \gamma_{n,i} d(w_{n,i}, T_i p) \\
&\leq \gamma_{n,0} d(x_n, p) + \sum_{i=1}^{N} \gamma_{n,i} H(T_i x_n, T_i p) \\
&\leq \gamma_{n,0} d(x_n, p) + \sum_{i=1}^{N} \gamma_{n,i} d(x_n, p) \\
&= d(x_n, p),
\end{aligned}
$$

which leads to $d(y_n, p) \leq d(x_n, p)$, $d(z_n, p) \leq d(x_n, p)$, and $d(x_{n+1}, p) \leq d(x_n, p)$. Thus, we conclude that the $\lim_{n\to\infty} d(x_n, p)$ exist and the sequence $\{x_n\}$ is bounded. $\square$

Lemma 4.5. *Let $\{T_i\}_{i=1}^{N}$ be multivalued nonexpansive mappings from K to $C(X)$ with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, $T_i p = \{p\}$ for all $p \in F$. Let $\{x_n\}$ be a sequence defined by* (4.2). *Then, we have $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, N$.*

Proof. Let $p \in F$. By Lemma 4.4, the limit $\lim_{n\to\infty} d(x_n, p)$ exist and the sequence $\{x_n\}$ is bounded. Now, let $\lim_{n\to\infty} d(x_n, p) = c$. Since $d(y_n, p) \leq d(x_n, p)$ and $d(u_{n,i}, p) \leq d(y_n, p), \limsup_{n\to\infty} d(y_n, p) \leq c$ and $\limsup_{n\to\infty} d(u_{n,i}, p) \leq c$. By the same arguments, we obtain that $\limsup_{n\to\infty} d(z_n, p) \leq c$ and $\limsup_{n\to\infty} d(v_{n,i}, p) \leq c$, $\limsup_{n\to\infty} d(x_n, p) \leq c$, and $\limsup_{n\to\infty} d(w_{n,i}, p) \leq c$. Moreover, we have

$$c = \lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d\left(\bigoplus_{i=1}^{N} \lambda_{n,i} u_{n,i}, p\right) \leq \lim_{n\to\infty} \sum_{i=1}^{N} \lambda_{n,i} d(u_{n,i}, p),$$

i.e.,

$$c \leq \lim_{n\to\infty} \sum_{i=1}^{N} \lambda_{n,i} \limsup_{n\to\infty} d(u_{n,i}, p) \leq \lim_{n\to\infty} \sum_{i=1}^{N} \lambda_{n,i} c \leq c,$$

which gives

$$\lim_{n\to\infty} d\left(\bigoplus_{i=1}^{N} \lambda_{n,i} u_{n,i}, p\right) = c.$$

It follows from Lemma 4.3 that $\lim_{n\to\infty} d(u_{n,i}, u_{n,j}) = 0$ for all $i, j = 1, 2, \ldots, N$. Therefore, we get

$$\begin{aligned} d(x_{n+1}, p) = d\left(\bigoplus_{i=1}^{N} \lambda_{n,i} u_{n,i}, p\right) &\leq \sum_{i=1}^{N} \lambda_{n,i} d(u_{n,i}, p) \\ &\leq \sum_{i=1}^{N} \lambda_{n,i} \big[d(u_{n,i}, u_{n,m}) + d(u_{n,m}, p)\big] \\ &\leq d(u_{n,m}, p) + \sum_{i=1}^{N} \lambda_{n,i} d(u_{n,i}, u_{n,m}), \end{aligned}$$

which implies that $\liminf_{n\to\infty} d(u_{n,m}, p) \geq c$ for all $m = 1, 2, \ldots, N$.

Since $\limsup_{n\to\infty} d(u_{n,i,}, p) \leq c$ and $d(u_{n,i,}p) \leq d(y_n, p)$, $\lim_{n\to\infty} d(u_{n,i}, p) = \lim_{n\to\infty} d(y_n, p) = c$. On the other hand, we have

$$\begin{aligned} c = \lim_{n\to\infty} d(y_n, p) &= \lim_{n\to\infty} d\left(\bigoplus_{i=1}^{N} \alpha_{n,i} w_{n,i} \oplus \bigoplus_{i=1}^{N} \beta_{n,i} v_{n,i}, p\right) \\ &\leq \lim_{n\to\infty} \left[\sum_{i=1}^{N} \alpha_{n,i} \limsup_{n\to\infty} d(w_n, p) + \sum_{i=1}^{N} \beta_{n,i} \limsup_{n\to\infty} d(w_{n,i}, p)\right] \\ &\leq \lim_{n\to\infty} \left[\sum_{i=1}^{N} \alpha_{n,i} c + \sum_{i=1}^{N} \beta_{n,i} c\right] \leq c, \end{aligned}$$

which yields

$$\lim_{n\to\infty} d\left(\bigoplus_{i=1}^{N} \alpha_{n,i} w_{n,i} \oplus \bigoplus_{i=1}^{N} \beta_{n,i} v_{n,i}, p\right) = c.$$

By Lemma 4.5, we have

$$\lim_{n\to\infty} d(v_{n,i}, v_{n,j}) = \lim_{n\to\infty} d(v_{n,i}, w_{n,j}) = \lim_{n\to\infty} d(w_{n,i}, w_{n,j}) = 0 \tag{4.3}$$

for all $i, j = 1, 2, \ldots, N$. Now, we have

$$\begin{aligned} d(y_n, p) &= d\left(\bigoplus_{i=1}^{N} \alpha_{n,i} w_{n,i} \oplus \bigoplus_{i=1}^{N} \beta_{n,i} v_{n,i}, p\right) \\ &\leq \sum_{i=1}^{N} \alpha_{n,i} d(w_{n,i}, p) + \sum_{i=1}^{N} \beta_{n,i} d(v_{n,i}, p) \\ &\leq \sum_{i=1}^{N} \alpha_{n,i} \big[d(w_{n,i}, v_{n,m}) + d(v_{n,m}, p)\big] + \sum_{i=1}^{N} \beta_{n,i} d(v_{n,i}, p) \\ &\leq \sum_{i=1}^{N} \alpha_{n,i} d(w_{n,i}, v_{n,m}) + \left(1 - \sum_{i=1}^{N} \beta_{n,i}\right) d(v_{n,m}, p) + \sum_{i=1}^{N} \beta_{n,i} d(v_{n,i}, p) \\ &= \sum_{i=1}^{N} \alpha_{n,i} d(w_{n,i}, v_{n,m}) + d(v_{n,m}, p) + \sum_{i=1}^{N} \beta_{n,i} \big[d(v_{n,i}, p) - d(v_{n,m}, p)\big] \\ &\leq \sum_{i=1}^{N} \alpha_{n,i} d(w_{n,i}, v_{n,m}) + d(v_{n,m}, p) \\ &\quad + \sum_{i=1}^{N} \beta_{n,i} \big[d(v_{n,i}, v_{n,m}) + (d(v_{n,m}, p)) - d(v_{n,m}, p)\big], \end{aligned}$$

i.e.,

$$d(y_n,p) \le \sum_{i=1}^{N} \alpha_{n,i} d(w_{n,i}, v_{n,m}) + d(v_{n,m}, p) + \sum_{i=1}^{N} \beta_{n,i} d(v_{n,i}, v_{n,m}). \tag{4.4}$$

Estimates (4.3) and (4.4) yield $\liminf_{n\to\infty} d(v_{n,m}, p) \ge c$ for all $m = 1, 2, \ldots, N$. Since $\limsup_{n\to\infty} d(v_{n,i,}, p) \le c$ and $d(v_{n,i,} p) \le d(z_n, p)$,

$$\lim_{n\to\infty} d(v_{n,i}, p) = \lim_{n\to\infty} d(z_n, p) = c.$$

Now, we have

$$\begin{aligned}
c &= \lim_{n\to\infty} d(z_n, p) \\
&= \lim_{n\to\infty} d\left(\gamma_{n,0} x_n \oplus \bigoplus_{i=1}^{N} \gamma_{n,i} w_{n,i}, p\right) \\
&\le \lim_{n\to\infty} \left[\gamma_{n,0} \limsup_{n\to\infty} d(x_n, p) + \sum_{i=1}^{N} \gamma_{n,i} \limsup_{n\to\infty} d(w_{n,i}, p)\right] \\
&\le \lim_{n\to\infty} \left[\gamma_{n,0} c + \sum_{i=1}^{N} \gamma_{n,i} c\right] \le c,
\end{aligned}$$

which implies

$$\lim_{n\to\infty} d\left(\gamma_{n,0} x_n \oplus \bigoplus_{i=1}^{N} \gamma_{n,i} w_{n,i}, p\right) = c.$$

As $\limsup_{n\to\infty} d(x_n, p) \le c$ and $\limsup_{n\to\infty} d(w_{n,i}, p) \le c$, it follows from Lemma 4.3 that $\lim_{n\to\infty} d(x_n, w_{n,i}) = \lim_{n\to\infty} d(w_{n,i}, w_{n,j}) = 0$ for all $i, j = 1, 2, \ldots, N$. Hence, $d(x_n, T_i x_n) \le d(x_n, w_{n,i})$ and $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, N$. □

Theorem 4.6. *Let $\{T_i\}_{i=1}^N$ be multivalued nonexpansive mappings from K to $KC(X)$ with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, $T_i p = \{p\}$ for all $p \in F$. Let $\{x_n\}$ be a sequence defined by* (4.2). *Then, $\{x_n\}$ Δ-converges to $p \in F$ in which (p,p) is a solution of the problem* (4.1).

Proof. It follows from Lemmas 4.4 and 4.5 that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$ and $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i \in \{1, 2, ..., N\}$, respectively. Let $\omega_w(x_n) := \cup A(\{u_n\})$ in which union is taken on all subsequences $\{u_n\}$ of $\{x_n\}$. For Δ-convergency of $\{x_n\}$, it is enough to show that $\omega_w(x_n) \subseteq F$ and $\omega_w(x_n)$ contains single point.

By Lemma 1.3, we have $\omega_w(x_n) \subset K$. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A\{u_n\} = \{u\}$. By Lemmas 1.3 and 1.4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$, which $\Delta-$convergent to v. Let fix $i \in \{1, 2, \dots, N\}$. As $T_i v$ is compact, then one can choose $z_{n,i} \in T_i v$ with $d(v_n, z_{n,i}) = d(v_n, T_i v)$ for all $n \geq 1$. Also, by the compactness of $T_i v$, there exists a convergent subsequence $\{z_{n_k,i}\}$ of $\{z_{n,i}\}$ such that $z_{n_k,i} \to w_i \in T_i v$. By the nonexpansiveness of T_i, we have

$$\begin{aligned} d(v_{n_k}, z_{n_k,i}) = d(v_{n_k}, T_i v) &\leq d(v_{n_k}, T_i v_{n_k}) + H(T_i v_{n_k}, T_i v) \\ &\leq d(v_{n_k}, T_i v_{n_k}) + d(v_{n_k}, v). \end{aligned}$$

Thus,

$$d(v_{n_k} w_i) \leq d(v_{n_k}, z_{n_k,i}) + d(z_{n_k,i}, w_i) \leq d(v_{n_k}, T_i v_{n_k}) + d(v_{n_k}, v) + d(z_{n_k,i}, w_i),$$

which further yieldslim $\sup_{n\to\infty} d(v_{n_k}, w_i) \leq \limsup_{n\to\infty} d(v_{n_k} v)$.

By the uniqueness of the asymptotic centers, we have $w_i = v \in T_i v$. As $i \in \{1, 2, \dots, N\}$ is arbitrary, we have $v \in F = \bigcap_{i=1}^N F(T_i)$. By Lemmas 4.4 and 1.4, $\lim_{n\to\infty} d(x_n, v)$ exists and $u = v \in F$. Thus, we have $\omega_w(x_n) \subseteq F$.

Consider a subsequence $\{u_n\}$ of $\{x_n\}$ with $A\{u_n\} = \{u\}$ and $A\{x_n\} = \{x\}$. Since $u \in \omega_w(x_n) \subseteq F$ and $\lim_{n\to\infty} d(x_n, v)$ exist, we have $u = x$ by Lemma 1.4. □

Theorem 4.7. *If K is compact and $\{T_i\}_{i=1}^N$ are multivalued nonexpansive mappings from K to $C(X)$ with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, $T_i p = \{p\}$ for all $p \in F$, then the sequence $\{x_n\}$ defined by* (4.2) *strongly converges to $p \in F$ in which (p, p) is a solution of problem* (4.1).

Proof. By Lemmas 4.4 and 4.5, we have that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$ and $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i \in \{1, 2, \dots, N\}$. Since K is compact, there is a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $\lim_{i\to\infty} x_{n_k} = q$. Then, for all $i \in \{1, 2, \dots, N\}$, we have

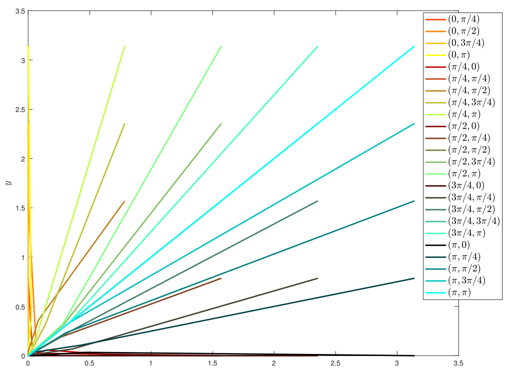
$$\begin{aligned} d(q, T_i q) &\leq d(q, x_{n_k}) + d(x_{n_k}, T_i x_{n_k}) + H(T_i x_{n_k}, T_i q) \\ &\leq d(q, x_{n_k}) + d(x_{n_k}, T_i x_{n_k}) + d(x_{n_k}, q). \end{aligned} \tag{4.5}$$

By passing to the limit on k in (4.5), we obtain $q \in T_i q$ for all $i \in \{1, 2, \dots, N\}$. Hence $p \in F$. □

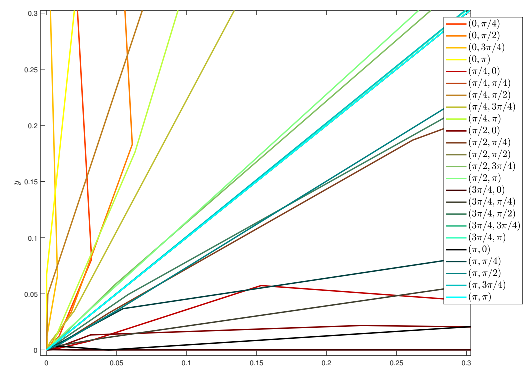
Example 4.8. Let $X = \mathbb{R}^2$ and $K_n = [0,\pi]^2$ for all $n \in \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$. Define $T_n : K_n \to KC(X)$ by

$$T_1(x,y) = B_{\frac{|x-y|}{2\sqrt{2}}}\left[\frac{x}{2}, \frac{y}{2}\right], \quad T_n(x,y) = B_{\frac{|x-y|}{2\sqrt{2}}}\left[\frac{\sin x}{n}, \frac{\sin y}{n}\right], \quad \text{if } \ n > 1.$$

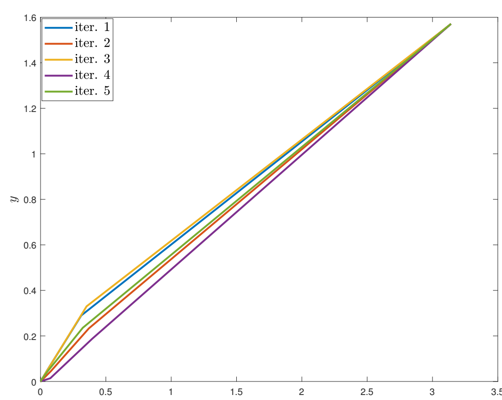
Then, T_n for all $n \in \{1, 2, \ldots, N\}$ is a nonexpansive mapping and has a fixed point $(0,0) \in T_n(0,0) = \{(0,0)\}$. Therefore, $\{(0,0),(0,0)\}$ is a solution of problem (4.1). The convergence behaviours of iteration scheme (4.2) for different choice of initial points for $N = 3$ are shown in Figure 2.



(A) Graphs in $[0,\pi]^2$



(B) (A) zoomed in $[0,0.3]^2$



(C)

FIGURE 2. The convergence behaviours of iteration scheme (4.2) for different choice of initial points (A-B) and for the same initial point (C).

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6. Declarations

Conflicts of Interest/ Competing Interests. The authors have no conflicts of interest to declare that are relevant to the content of this article.

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